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NON-STATIONARY HEAT TRANSFER IN TWO-PHASE HOLLOW CYLINDER WITH FUNCTIONALLY GRADED EFFECTIVE MATERIAL PROPERTIES WITH SECOND KIND OF BOUNDARY CONDITIONS

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The unidirectional non-stationary heat conduction in two-phase hollow cylinder is considered. The conductor is made of two-phase stratified composites and has a smooth gradation of effective properties in the radial direction. Therefore, we deal here with a special case of functionally graded materials, FGM (cf. [6]). The formulation of mathematical model of the conductor is based on the tolerance averaging approach (TAA), cf. [8]. Considerations in this paper are restricted only to the unidirectional non-stationary heat conduction, where on the boundaries are given φ-constant or periodic function of heat fluxes $q_0 = q_0(\varphi,t)$, $q_k = q_k(\varphi,t)$ for every time period t, and function of initial temperature $\Theta^0 = \Theta^0(\varphi,t)$ for $t = t_0$. The effect of fibres width on the temperature field will be also examined.

Keywords: Heat conduction, FGM, composite

1. BASIC CONCEPTS

1.1. Subject of contribution

The main aim of this paper is to consider the heat conduction in two-phase hollow cylinder. This consideration concerns only with the non-stationary heat transfer problem in two-phase composite with a deterministic microstructure, which is, for a fixed radius ρ, periodic along the angular axis and has smooth and functional effective properties in the radial direction (Fig.1). Therefore, we deal here with a special case of functionally graded materials, FGM, cf. [6].

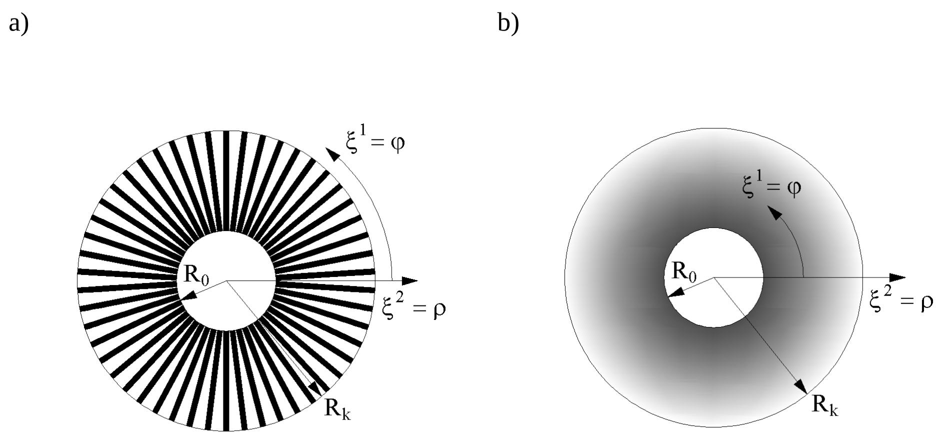

Fig. 1. Structure of the two-phase functionally graded composite in a) micro- and b) macro-scale

The analysis of the heat transfer in the hollow cylinder made from functionally graded materials we can find in [2], [5], where material properties are expressed as power or exponential functions of the radial coordinate. In paper [1] we can find application of higher-order theory for thermal analysis in functionally graded materials.

1.2. Model equations

The physical phenomenon of the non-stationary heat transfer is described by well known Fourier equation

$$c\dot{\Theta} - \nabla \cdot (\boldsymbol{k} \cdot \nabla\Theta) = Q_v \,, \tag{1.1}$$

which contains (in this case) highly oscillating and discontinuous coefficients $\boldsymbol{k}$ - heat conduction tensor, and c - specific heat, Q_v - internal thermal sources. The modelling problem is how to describe microheterogeneous conductor by certain averaged equations. The formulation of the macroscopic mathematical model for the analysis of heat transfer in the conductor under consideration will be based on the tolerance averaging technique, cf. [7-8]. The general description of this technique and application to analysis of longitudinally graded stratified media can be found in [4], [7].

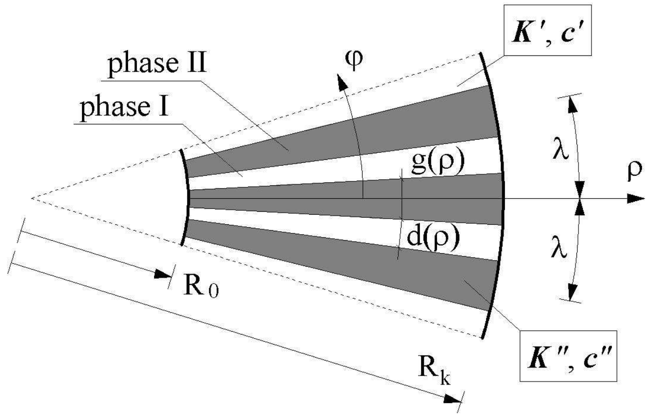

Fig. 2. Deterministic microstructure of composite

The object of our considerations is a hollow conductor with microstructure given on Fig.2. This microstructure is determined by the unit cell Δ with the diameter of $\lambda = 2\pi / N$, where N is a number of cells in considered composite.

The fibres width g is in general given by

$$g(\rho) = \eta\lambda R_0, \tag{1.2}$$

for $\eta \in [0,1)$, which implies functional macroscopic material properties in ρ-direction. Volume fractions of homogeneous layers are denoted by $\nu'(\rho) = d(\rho)/\lambda\rho$ and $\nu''(\rho) = g(\rho)/\lambda\rho$, where $d(\rho) = \lambda\rho - g(\rho)$. Dimensionless function $\nu = \sqrt{\nu'\nu''}$ is referred to as the distribution of heterogeneity.

The one of the fundamental assumptions in tolerance averaging approach concerns with the temperature field decomposition

$$\Theta(\varphi,\rho,t) = \theta(\varphi,\rho,t) + h(\varphi,\rho)\cdot\psi(\varphi,\rho,t), \tag{1.3}$$

where $\varphi \in [0,2\pi)$, $\rho \in [R_0, R_k]$ and $t \geq 0$. Functions of averaged temperature θ and oscillation amplitude temperature ψ are assumed to be slowly varying, i.e. $\theta(\cdot,\rho,t),\psi(\cdot,\rho,t) \in SV_\delta^1(\Omega,\Delta)$. The exact definition of the *slowly varying* and *tolerance periodic* function can be found in [7-8]. The expected form of the temperature oscillations, caused by discontinuity of the coefficients in (1.1), is assured by the "saw-type" *locally periodic* function (Fig.3), which would be called the fluctuation shape function h

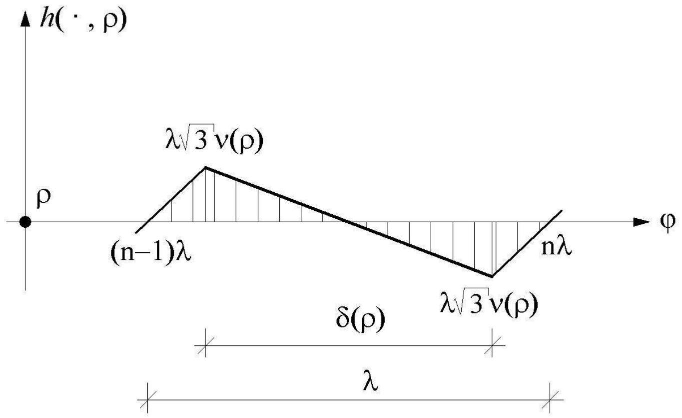

Fig. 3. Fluctuation shape function

where $\delta(\rho) = d(\rho)/\rho$.

The second concept of the modelling technique is the averaging operation

$$\langle f \rangle(\varphi,\rho) = \frac{1}{|\Delta|} \int_{\varphi-\lambda/2}^{\varphi+\lambda/2} f(z,\rho)dz\,, \tag{1.4}$$

where $|\Delta| = \lambda$. On the grounds of this definition we can formulate the second modelling assumption, the tolerance averaging approximation. In the course of modelling it is assumed that terms $O(\varepsilon)$ are negligibly small, where ε is a certain tolerance parameter, cf. [7]. For the arbitrary tolerance periodic function $f \in TP^1_\varepsilon(\Omega,\Delta)$, slowly varying function $F \in SV^1_\varepsilon(\Omega,\Delta)$ and fluctuation shape function $h \in FS^1_\varepsilon(\Omega,\Delta)$ we have

$$\begin{bmatrix} \langle fF \rangle = \langle f \rangle F + O(\varepsilon) \\ \langle f\nabla(hF) \rangle = \langle f\partial h \rangle F + \langle fh \rangle \overline{\nabla} F + O(\varepsilon) \end{bmatrix}, \tag{1.5}$$

tolerance model. Bearing in mind the mean value definition (1.4) and all model assumptions, we conclude to the system of averaged equations (cf. [7]):

$$\begin{cases} \nabla \cdot \left(\langle \boldsymbol{k} \rangle \nabla \theta + \langle \boldsymbol{k}\partial h \rangle \psi \right) - \langle c \rangle \dot{\theta} = 0 \\ \overline{\nabla} \cdot \left(\langle \boldsymbol{k} h^2 \rangle \overline{\nabla}\, \psi \right) - \langle \boldsymbol{k}\partial h \rangle \nabla \theta - \langle \boldsymbol{k}\partial h^2 \rangle \psi - \langle c h^2 \rangle \dot{\psi} = 0 \end{cases}, \tag{1.6}$$

describing two dimensional heat conduction in two-phase hollow cylinder, where internal thermal sources Q_v are neglected. The coefficients

$$\begin{bmatrix} \langle \boldsymbol{k} \rangle = k'\nu' + k''\nu'' \\ \langle \boldsymbol{k} h^2 \rangle = \lambda^2 \nu^2 \langle \boldsymbol{k} \rangle \end{bmatrix}, \begin{bmatrix} \langle \boldsymbol{k} \partial h \rangle = 2\nu\sqrt{3}(k' - k'') \\ \langle \boldsymbol{k} \partial h^2 \rangle = 12(k'\nu'' + k''\nu') \end{bmatrix}, \begin{bmatrix} \langle c \rangle = c'\nu' + c''\nu'' \\ \langle c h^2 \rangle = \lambda^2 \nu^2 \langle c \rangle \end{bmatrix}, \tag{1.7}$$

are continues and functional. The gradient operators in the above equations have the form

$$\nabla = \left(\frac{\partial}{\partial \varphi}, \frac{\partial}{\partial \rho} \right), \ \partial = \left(\frac{\partial}{\partial \varphi}, 0 \right), \ \overline{\nabla} = \left(0, \frac{\partial}{\partial \rho} \right). \tag{1.8}$$

The obtained averaged differential equations (1.6) have smooth functional coefficients in contrast to coefficients in equation (1.1). To obtain the results, numerical methods had to be used (Maple software in this case). This model takes into account an effect of microstructure size on the overall heat transfer behaviour.

2. EXAMPLES OF APPLICATION

The main aim of this chapter is to display mostly an effect of parameter η in (1.2) on the temperature field in time. These considerations concern the unidirectional heat transfer for a two-phase conductor with deterministic microstructure (Fig.2) of $N = 60$ cells for geometric values of $R_0 = 1\,[m]$ and $R_k = 3\,[m]$. In general we shall denote isotropic tensor of conductivity for each of components

$$\boldsymbol{k} = k \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{2.1}$$

where fixed values of conductivity are listed below.

Table 1. Material properties

	phase I	phase II
$c\,[Jm^{-3}K^{-1}]$	3432000	14600
$k\,[Wm^{-1}K^{-1}]$	58	0.045

Because of the decomposition (1.3), twice more initial-boundary conditions are needed as distinct from deterministic problem approach. Hence, in the interior of considered composite we denote the initial temperature $\Theta^0(t = 0) = 0\,[^0C]$ as:

$$\theta^0 = \left\langle \Theta^0 \right\rangle = 0 \left[{}^0C \right] \text{ and } \psi^0 = \left\langle \Theta^0 h \right\rangle = 0 \left[{}^0C \right]. \tag{2.2}$$

The second kind of boundary condition we shall understand here as a fixed heat flux value on the boundary. Moreover, for the heat fluxes in the radial direction, q_0 and q_k, on the inner and outer boundary, respectively, we calculate

$$\begin{bmatrix} \overline{\nabla}\,\theta(R_0) = -\dfrac{\langle q_0 \rangle}{\langle \boldsymbol{k} \rangle} \\ \overline{\nabla}\,\psi(R_0) = -\dfrac{\langle q_0 h \rangle}{\langle \boldsymbol{k} h^2 \rangle} \end{bmatrix} \text{ and } \begin{bmatrix} \overline{\nabla}\,\theta(R_k) = -\dfrac{\langle q_k \rangle}{\langle \boldsymbol{k} \rangle} \\ \overline{\nabla}\,\psi(R_k) = -\dfrac{\langle q_k h \rangle}{\langle \boldsymbol{k} h^2 \rangle} \end{bmatrix}. \tag{2.3}$$

If F is a φ-constant function, then $\langle Fh \rangle = 0$. All above conditions and formulations will be used for all following examples in subsequent part of this paper.

2.1. Heat transfer in time

Let us consider two-phase hollow cylinder (Fig.1) under heat fluxes $q_0(R_0) = 1000 \left[Wm^{-2} \right]$ and $q_k(R_k) = 0 \left[Wm^{-2} \right]$, where fibres width is expressed by (1.2) for $\eta = 0.5$.

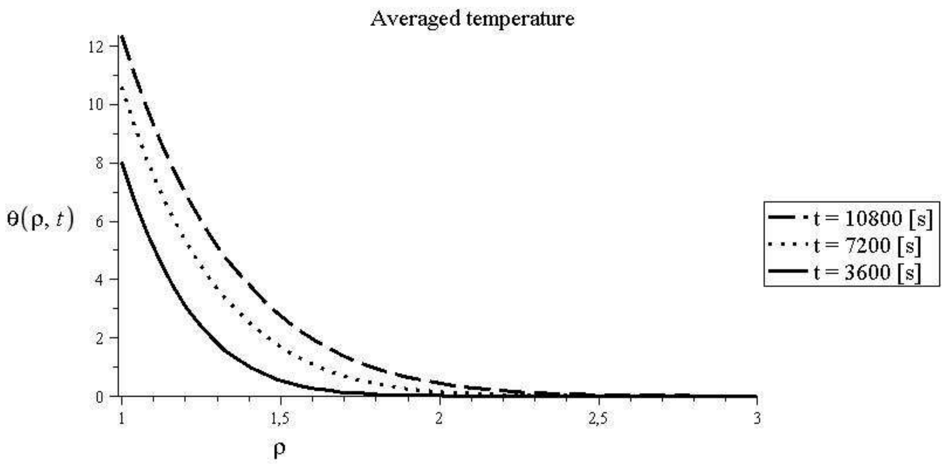

Fig. 4. Averaged temperature varying in time

Since the averaged temperature vary in time, the amplitude oscillation temperature is constant and equal to zero for every time *t*.

2.2. Fibres width effect

Various values of parameter η are considered and theirs effect on temperature field. Initial-boundary conditions are the same as in 2.1. The obtained results after t = 3600 [s] for averaged temperature:

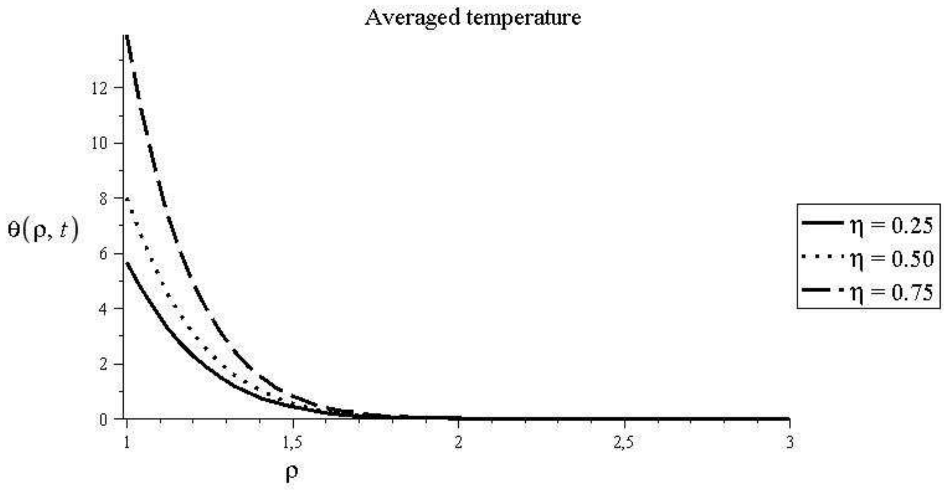

Fig. 5. Fibres width effect on averaged temperature after one hour

and amplitude oscillation temperature is equal to zero. The wider fibres, the faster decreasing of temperature near the inner boundary, but the greater value of the temperature on the inner surface of considered conductor.

2.3. Periodic load effect

The last example deals with particular case, which is very similar to that from 2.1. Geometry and initial-boundary conditions are the same, except the heat flux on the inner boundary, which is not constant but periodic along angular axis:

$$q_0(\varphi) = \left[1 + sin\left(\frac{\pi\varphi}{\lambda}\right)\right] \cdot 1000 \left[Wm^{-2}\right] \qquad (2.4)$$

Since $\langle q_0 \rangle = 1000 \left[Wm^{-2}\right]$, the averaged temperature course of time:

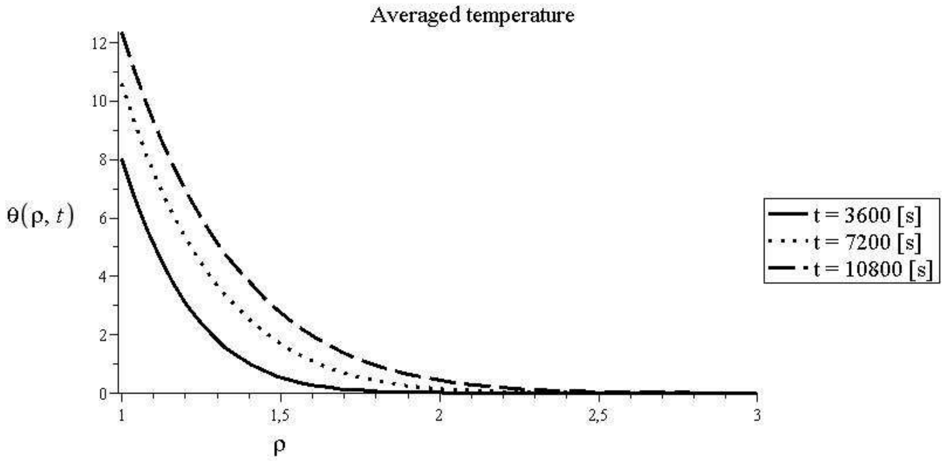

Fig. 6. Averaged temperature varying in time

is the same as that on Fig.4. However, this time the amplitude oscillation temperature near the inner boundary is not equal to zero:

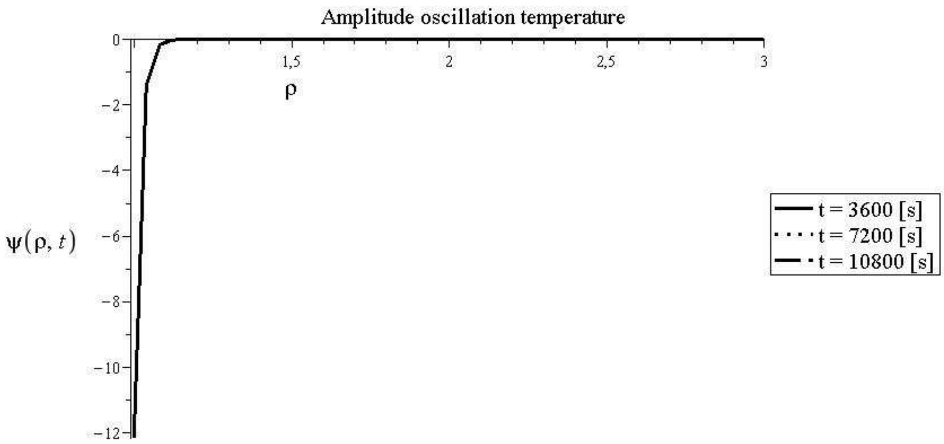

Fig. 7. Amplitude oscillation temperature varying in time

The diagram on Fig.7 reveals that ψ-function does not vary in time.

3. SUMMARY

The tolerance averaging approximation leads to the mathematical model of composites conductor with functionally graded material properties. The obtained model equations have continuoues coefficients in opposition to a discrete model, where they are strongly oscillating. Since the proposed model equations have smooth functional coefficients then in most cases solutions to specific problem, for heat conductor under consideration, have to be obtained using well known numerical methods (Maple software). The tolerance model takes into account an effect of the microstructure size on the temperature field. Moreover, by changing fibres width, we can obtain desirable temperature field inside composite. However, for the φ-constant boundary conditions and isotropic material properties for each of components, there is no temperature oscillation revealed.

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NIESTACJONARNY PRZEPŁYW CIEPŁA W PRZEWODNIKU CYLINDRYCZNYM Z WARUNKAMI BRZEGOWYMI DRUGIEGO RODZAJU

Streszczenie

Przedmiotem rozważań niniejszej pracy jest analiza wpływu pewnych parametrów geometrycznych przewodnika na przebieg pola temperatur dla zagadnienia niestacjonarnego przepływu ciepła. Rozpatrywany przewodnik jest dwuskładnikowym kompozytem o deterministycznej mikrostrukturze, który w kierunku kątowym jest λ-periodyczny (dla ustalonego promienia ρ), a w kierunku promieniowym jego efektywne własności zmieniają się w sposób wolnozmienny. Stąd, mamy tutaj do czynienia ze szczególnym przypadkiem materiału o funkcyjnej gradacji własności, FGM (por. Suresh , Mortensen, 1998). Samo zjawisko przewodznia ciepła opisane jest równaniem Fouriera, które zawiera nieciągłe i silnie oscylujące współczynniki. Model matematyczny opisujący zjawisko przewodzenia ciepła w rozpatrywanym kompozycie opierać się będzie na technice tolerancyjnej aproksymacji (por. Woźniak, Wierzbicki, 2000). W pracy ograniczymy się jedynie do przypadku jednowymiarowego przepływu ciepła, w którym na brzegach przewodnika dana jest stała bądź periodyczna funkcja gęstości strumienia ciepła $q_0 = q_0(\varphi,t)$, $q_k = q_k(\varphi,t)$ dla dowolnej chwili czasu t, oraz funkcja pola temperatury $\Theta^0 = \Theta^0(\varphi,t)$ w chwili początkowej $t = t_0$. Rozpatrywano również wpływ szerokości inkluzji na prędkość zmian pola temperatury w obszarze przewodnika.