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RELIABILITY DESIGN OF COMPLEX SYSTEMS BY MINIMIZING THE LIFETIME VARIANCE

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The problem considered in this paper is the minimization of the lifetime variance of a complex system subject to its expected life and economic constraints. The example of a bridge network, in which all elements have constant failure rates, illustrates the problem. A numerical algorithm for solving this optimization problem by using exact formulae for system lifetime moments is included. Using this algorithm, we can obtain results better than the solutions known from earlier papers.

Keywords: reliability, constant failure rate, redundancy

1. Introduction

It is well known that redundant elements increase system reliability. A non-trivial question arises then: How to optimally allocate redundant elements? The answer depends on the criterion of optimality and on the structure of the designed system. Systems most extensively studied in the literature are those with a series-parallel structure. For these systems there are algorithms which maximize system reliability (Prasad and Raghavachari, 1998; Prasad *et al.*, 1999), determine the minimal cost (Levitin *et al.*, 1998), or optimize some objective function (Coit and Smith, 1996). Procedures for a wide class of systems which maximize a lower percentile of the system time-to-failure distribution or maximize the reliability of systems subjected to imperfect fault-coverage are described in (Coit and Smith, 1998) and (Amari *et al.*, 1999), respectively. The proposed design algorithms are illustrated with numerical examples of a rather high complexity.

From a practical point of view, in some situations it is important to find parameters which ensure a fixed expected system lifetime. Of course, many such solutions exist and the ones which minimize the variance of the system lifetime may be regarded as optimal. This is because a smaller variance guarantees that a real system lifetime is better estimated by the expected lifetime. This is one of the possible approaches to the reliability optimization problem and its detailed description is presented in (Krishnan Iyer and Downs, 1977; 1978).

Although the formulation of the optimization problem is simple, it is very difficult to find an exact solution. Even in the case where constant failure rates of all units are assumed, we encounter a non-linear mixed integer programming problem of a high dimension, difficult to solve explicitly.

The method of solving the optimization problem presented in (Krishnan Iyer and Downs, 1978) uses formulae approximating the first and second moments of parallel systems (the authors determine complex system reliability by minimal cutsets). These formulae are heuristic and they significantly simplify the computations. However, the region of their validity is limited and it is necessary to check the domain of optimization every time the formulae are used.

In this paper the system is characterized by minimal paths, so we can solve the optimization problem by employing exact formulae for moments of series-parallel systems. The procedure is applied to the numerical example given in (Krishnan Iyer and Downs, 1978) in order to compare the results. The comparison of the results obtained using both exact and approximate formulae gives useful information applicable to solving similar problems. If these results turn out to be comparable, then it is convenient to use approximate formulae because algorithms based on them require fewer computations. On the contrary, if an exact method leads to significantly better results, it can be profitable to perform more complex computations to increase the system's performance.

1.1. System Description

All the deliberations in this paper can be applied to any complex system. For convenience, we concentrate on the bridge network U shown in Fig. 1. System U consists of five subsystems U_i ($i = 1, \ldots, 5$), where every U_i is a subsystem with n_i units in parallel. The terms 'parallel' and 'series' are used in their diagram-logic sense. All units work independently of one another and have constant failure rates λ_i, which are common for all units in one subsystem. Thus the system may be identified by two vectors: $\underline{n} = (n_1, \ldots, n_5)^{\mathrm{T}}$ and $\underline{\lambda} = (\lambda_1, \ldots, \lambda_5)^{\mathrm{T}}$. The units U_3 and U_4 act as alternatives to U_1 and U_2, and adding U_5 results in increasing system reliability. It is reasonable to answer the question concerning the numbers and characteristics of redundant elements in the respective units.

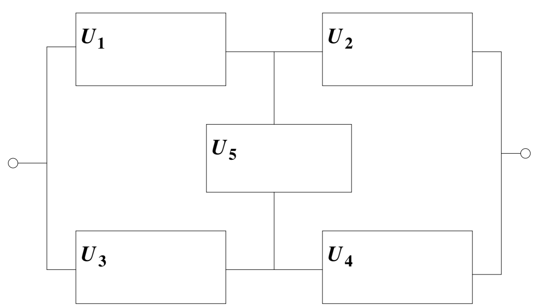

Fig. 1. Bridge network.

The reliability of complex systems can be described in terms of 'minimal success paths' or 'minimal cutsets'. For the bridge network U there are four minimal paths: $p_1 = \{U_1, U_2\}$, $p_2 = \{U_3, U_4\}$, $p_3 = \{U_1, U_4, U_5\}$, $p_4 = \{U_2, U_3, U_5\}$, and four minimal cutsets: $c_1 = \{U_1, U_3\}$, $c_2 = \{U_2, U_4\}$, $c_3 = \{U_1, U_4, U_5\}$, $c_4 = \{U_2, U_3, U_5\}$. Using the minimal paths, we can calculate the reliability of system U as

$$\begin{aligned} R_S(t) &= R_{\underline{12}}(t) + R_{\underline{34}}(t) + R_{\underline{145}}(t) + R_{\underline{235}}(t) \\ &\quad - R_{\underline{1234}}(t) - R_{\underline{1245}}(t) - R_{\underline{1235}}(t) \\ &\quad - R_{\underline{1345}}(t) - R_{\underline{2345}}(t) + 2R_{\underline{12345}}(t), \quad (1) \end{aligned}$$

where $R_{\underline{ij}}(t)$ denotes the reliability of a system built from U_i and U_j in series. If minimal cutsets are used, we have

$$\begin{aligned} R_P(t) &= R_{|13|}(t) + R_{|24|}(t) + R_{|145|}(t) + R_{|235|}(t) \\ &\quad - R_{|1234|}(t) - R_{|1245|}(t) - R_{|1235|}(t) \\ &\quad - R_{|1345|}(t) - R_{|2345|}(t) + 2R_{|12345|}(t), \quad (2) \end{aligned}$$

where $R_{|ij|}(t)$ denotes the reliability of a system built from U_i and U_j in parallel. Of course, $R_S(t) = R_P(t)$ for every $t \in \mathbb{R}$. For any complex system its reliability is a linear combination of reliabilities of some parallel (if cutsets are employed) or series-parallel systems (if paths are employed).

1.2. System Optimization Problem

Let the random variable T denote the lifetime of system U. We wish to obtain the values of system parameters which ensure that the mean system lifetime will be equal to some fixed value and the variance will be minimal. Of course, according to economic conditions, in any real system some constraints must be satisfied. Any unit in subsystem U_i determines the cost function $c(\lambda_i) = a_i/(b_i - 1/\lambda_i)$, where a_i and b_i are fixed cost parameters and $1/\lambda_i$ is the expected lifetime of a single element. Function $c(\lambda_i)$ has two asymptotes (see Fig. 2): the vertical and horizontal one, representing technological limitations and the minimal cost, respectively. System cost is defined by the sum of all component costs: $c_{\mathrm{syst}} = \sum_{i=1}^{5} n_i c(\lambda_i)$. The more elements there are in the system and the longer their lifetime, the higher the cost of the system.

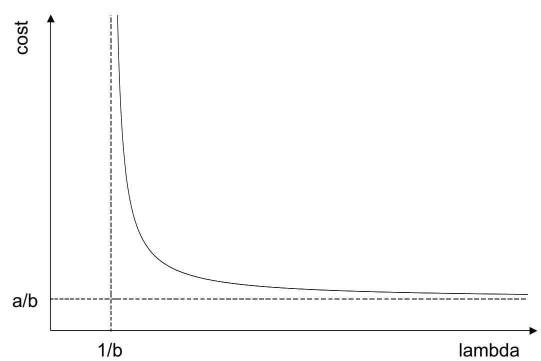

Fig. 2. Cost function.

The optimization problem is formulated as follows: Find vectors $\underline{n}_{\mathrm{opt}}$ and $\underline{\lambda}_{opt}$ which minimize the variance of the system lifetime, subject to its expected life and economic constraints:

$$\begin{cases} (\underline{n}_{\mathrm{opt}}, \underline{\lambda}_{\mathrm{opt}}) = \arg\min \operatorname{Var} T, \\ \mathrm{E}\, T = \mathrm{E}_0, \\ c_{\mathrm{syst}} \le c_0. \end{cases}$$

Notice that here minimizing the variance is equivalent to minimizing the second moment of T (because $\operatorname{Var} T = \mathrm{E}\, T^2 - (\mathrm{E}\, T)^2$ and $\mathrm{E}\, T$ is equal to a fixed value E_0). This is a non-linear mixed integer programming problem, in which additional difficulty is caused by the fact that discrete variables n_i appear as the upper limits of the sum in the performance index $\mathrm{E}\, T^2$ and in the constraint for $\mathrm{E}\, T$.

2. Exact Formulae for $\mathrm{E}\,T$ and $\mathrm{E}\,T^2$

Let the random variable T_{ij} denote the lifetime of the j-th element in U_i, and let T_i be the lifetime of subsystem U_i. Obviously, $T_i = \max_j T_{ij}$ and the reliability of U_i is equal to

$$R_i(t) = 1 - \Pr\{T < t\} = 1 - \prod_{j=1}^{n_i} \big(1 - R_{ij}(t)\big),$$

where $R_{ij} = \exp(-\lambda_i t)$. The reliability of the path $\{U_{l_1}, \dots, U_{l_k}\}$ is

$$R_{\underline{l_1 \dots l_k}}(t) = \prod_{x=1}^{k} R_{l_x}(t),$$

and the reliability of the cutset $\{U_{l_1}, \dots, U_{l_k}\}$ is

$$R_{|l_1,\dots,l_k|}(t) = 1 - \prod_{x=1}^{k} (1 - R_{l_x}(t)).$$

Because T is non-negative, we can calculate moments of T as

$$\begin{aligned} \mathrm{E}\,T &= \int_0^\infty R_S(t)\,\mathrm{d}t = \int_0^\infty R_P(t)\,\mathrm{d}t, \\ \mathrm{E}\,T^2 &= \int_0^\infty 2tR_S(t)\,\mathrm{d}t = \int_0^\infty 2tR_P(t)\,\mathrm{d}t. \end{aligned} \tag{3}$$

Using (3), we have formulae for path moments:

$$\begin{aligned} \mathrm{E}_{\,\underline{l_1 \dots l_k}} T &= \sum_{i_{l_1}=1}^{n_{l_1}} \cdots \sum_{i_{l_k}=1}^{n_{l_k}} (-1)^{i_{l_1}+\dots+i_{l_k}+k} \\ &\quad \times \frac{\binom{n_{l_1}}{i_{l_1}} \cdots \binom{n_{l_k}}{i_{l_k}}}{i_{l_1}\lambda_{l_1} + \dots + i_{l_k}\lambda_{l_k}}, \\ \mathrm{E}_{\,\underline{l_1 \dots l_k}} T^2 &= \sum_{i_{l_1}=1}^{n_{l_1}} \cdots \sum_{i_{l_k}=1}^{n_{l_k}} (-1)^{i_{l_1}+\dots+i_{l_k}+k} \\ &\quad \times \frac{2\binom{n_{l_1}}{i_{l_1}} \cdots \binom{n_{l_k}}{i_{l_k}}}{(i_{l_1}\lambda_{l_1} + \dots + i_{l_k}\lambda_{l_k})^2}, \end{aligned} \tag{4}$$

and for cutset moments:

$$\mathrm{E}_{\,|ij|} T = \sum_{n=1}^{n_i+n_j} \sum_{a=0}^{n} (-1)^{n+1} \frac{\binom{n_i}{a}\binom{n_j}{n-a}}{a\lambda_i + (n-a)\lambda_j},$$

$$\mathrm{E}_{\,|ij|} T^2 = \sum_{n=1}^{n_i+n_j} \sum_{a=0}^{n} (-1)^{n+1} \frac{2\binom{n_i}{a}\binom{n_j}{n-a}}{(a\lambda_i + (n-a)\lambda_j)^2},$$

$$\begin{aligned} \mathrm{E}_{\,|ijk|} T &= \sum_{n=1}^{n_i+n_j+n_k} \sum_{a=0}^{n} \sum_{b=0}^{n-a} (-1)^{n+1} \\ &\quad \times \frac{\binom{n_i}{a}\binom{n_j}{b}\binom{n_k}{n-a-b}}{a\lambda_i + b\lambda_j + (n-a-b)\lambda_k}, \end{aligned}$$

$$\begin{aligned} \mathrm{E}_{\,|ijk|} T^2 &= \sum_{n=1}^{n_i+n_j+n_k} \sum_{a=0}^{n} \sum_{b=0}^{n-a} (-1)^{n+1} \\ &\quad \times \frac{2\binom{n_i}{a}\binom{n_j}{b}\binom{n_k}{n-a-b}}{(a\lambda_i + b\lambda_j + (n-a-b)\lambda_k)^2}, \quad \text{etc.} \end{aligned} \tag{5}$$

Formulae (1)–(5) allow us to find the expected lifetime and the variance of system U (it is better to determine moments of U by minimal paths, because the sums in (4) have fewer terms than the sums in (5)). Notice that moments of system U are functions of vectors $\underline{n}$ and $\underline{\lambda}$: $\mathrm{E}\,T = \mathrm{E}\,T(\underline{n}, \underline{\lambda})$ and $\mathrm{E}\,T^2 = \mathrm{E}\,T^2(\underline{n}, \underline{\lambda})$.

Now, from (4) and (5) it follows that for any scalar $\alpha > 0$, we have

$$\begin{aligned} \mathrm{E}\,T(\underline{n}, \alpha\underline{\lambda}) &= \alpha^{-1}\,\mathrm{E}\,T(\underline{n}, \underline{\lambda}), \\ \mathrm{E}\,T^2(\underline{n}, \alpha\underline{\lambda}) &= \alpha^{-2}\,\mathrm{E}\,T^2(\underline{n}, \underline{\lambda}). \end{aligned} \tag{6}$$

These expressions are extensively used in the minimization algorithm described in the next section. They enable us to find moments $\mathrm{E}\,T(\underline{n}, \alpha\underline{\lambda})$ and $\mathrm{E}\,T^2(\underline{n}, \alpha\underline{\lambda})$ for any scalar α by rescaling the previously computed values $\mathrm{E}\,T(\underline{n}, \underline{\lambda})$ and $\mathrm{E}\,T^2(\underline{n}, \underline{\lambda})$, without using summation expressions.

3. Algorithm for Numerical Minimization

This section describes a method used in the numerical minimization of $\mathrm{E}\,T^2(\underline{n}, \underline{\lambda})$ subject to the constraints $\mathrm{E}\,T(\underline{n}, \underline{\lambda}) = E_0$ and $c_{\mathrm{syst}}(\underline{n}, \underline{\lambda}) \le c_0$. To solve this problem, a modified 'full search' method was used. The optimization procedure consists of two stages. At the first stage the set of values of $\underline{n}$ is determined, for which there exists any $\underline{\lambda}$ satisfying the cost constraint. At the second stage, for each $\underline{n}$ the set of $\underline{\lambda}$ is determined, which represents the set of directions in $(\mathbb{R}^+)^5$. For fixed $\underline{\lambda}$, the value of $\mathrm{E}\,T(\underline{n}, \underline{\lambda})$ is calculated. Using (6), a vector $\underline{\lambda}'$ is determined for which $\mathrm{E}\,T(\underline{n}, \underline{\lambda}') = E_0$. The values of $\underline{\lambda}'$ which do not satisfy the cost constraint are discarded. Next, the minimum value of $\mathrm{E}\,T^2(\underline{n}, \underline{\lambda}')$ is evaluated for each $\underline{n}$. A global minimum is obtained by comparing the minima for all $\underline{n}$.

The detailed description of the minimization procedure is as follows:

- The first stage:

  1. Find $M_i = \max\{m : c_{\min}(\underline{n}_{i,m}) \le c_0\}$ for $i = 1, \dots, 5$, where

$$(\underline{n}_{i,m})_j = \begin{cases} 1 & \text{for} \quad i \ne j, \\ m & \text{for} \quad i = j, \end{cases}$$

and $c_{\min}(\underline{n}) = \sum_{i=1}^{5} n_i\, a_i/b_i$ denotes the minimal cost of subsystem U_i. M_i is the largest admissible number of elements in U_i.

2. Set $N = \{\underline{n} : 1 \le (\underline{n})_i \le M_i,\ i = 1, \ldots, 5\}$.

- The second stage:

1. Set $\Lambda = \{\epsilon_i + \eta_i/k_i : 0 \le k_i \le k,\ \sum_{i=1}^{5} k_i = k,\ i = 1, \ldots, 5\}$, where k is fixed and determines the number of mesh grids.
2. For each $\underline{n} \in N$ and each $\underline{\lambda} \in \Lambda$ compute $\mathrm{E}\,T(\underline{n}, \underline{\lambda})$.
3. Calculate $\underline{\lambda}' = \underline{\lambda}\mathrm{E}\,T(\underline{n}, \underline{\lambda})/\mathrm{E}_0$. From (6) we have

$$\mathrm{E}\,T(\underline{n}, \underline{\lambda}') = E_0,$$

$$\mathrm{E}\,T^2(\underline{n}, \underline{\lambda}') = \left(\frac{\mathrm{E}\,T(\underline{n}, \underline{\lambda})}{\mathrm{E}_0}\right)^2 \mathrm{E}\,T^2(\underline{n}, \underline{\lambda}).$$

4. If $\underline{\lambda}'$ does not satisfy the cost constraint (including technological limitations), discard it.
5. For a valid $\underline{\lambda}'$ compare the values of $\mathrm{E}\,T^2(\underline{n}, \underline{\lambda}')$ and find the minimum.
6. Compare the minima of $\mathrm{E}\,T^2$ for all $\underline{n}$.

4. Numerical Example

The minimization procedure described above was applied to the bridge network shown in Fig. 1. The assumed numerical values were the same as in (Krishnan Iyer and Downs, 1978): $\mathrm{E}_0 = 20$, $c_0 = 24.5$, $\underline{a} = (60, 95, 70, 55, 50)$, $\underline{b} = (45, 45, 45, 45, 33)$. The moments of T were evaluated using the minimal path approach. The results of the optimization procedure are presented in Tab. 1. It contains the optimal solution given by $\underline{n} = (1, 1, 4, 5, 1)$ (the 7th row) with $\mathrm{Var}\,T = 75.97$ and six solutions close to the optimal one.

Table 1. Variance minimization based on exact formulae.

$\underline{n}$	$\underline{\lambda}$	$\mathrm{Var}\,T$	cost
$(1,1,4,4,1)$	$(.0546, .0803, .0824, .0774, .0491)$	80.76	24.5
$(1,1,5,5,1)$	$(.1560, .2077, .0831, .0809, 1.979)$	80.32	24.5
$(2,1,4,4,1)$	$(.0851, .0554, .0832, .0769, .1247)$	79.01	24.5
$(2,1,3,5,1)$	$(.0761, .0680, .0673, .0922, .0735)$	78.03	24.5
$(1,1,3,6,1)$	$(.0566, .0822, .0591, .1023, .0738)$	77.58	24.5
$(1,1,5,4,1)$	$(.0656, .0812, .0904, .0725, .0962)$	77.06	24.5
$(1,1,4,5,1)$	$(.0601, .0809, .0765, .0886, .0749)$	75.97	24.5

In (Krishnan Iyer and Downs, 1978) the authors used the following formulae for the moments of parallel systems:

$$\mathrm{E}\,T \approx \frac{1}{\Lambda_H} \sum_{i=1}^{n} \frac{1}{i},$$

$$\mathrm{E}\,T^2 \approx \frac{1}{\Lambda_H^2} \sum_{i=1}^{n} \frac{1}{i^2} + \frac{1}{n}\left(\sum_{i=1}^{n} \frac{1}{i}\right)^2 \sum_{i=1}^{n} \frac{1}{\lambda_i^2},$$

where Λ_H stands for the harmonic mean of $\lambda_1, \ldots, \lambda_n$. In these formulae, if $(\max \lambda_i - \min \lambda_i) < 0.7\Lambda_H$, the approximation is within 5% of the exact value. Table 2 presents the results obtained in (Krishnan Iyer and Downs, 1978) using approximating formulae (the authors gave θ as the arithmetic mean of $\underline{\lambda}$ coordinates instead of the $\underline{\lambda}$ vector).

Table 2. Variance minimization based on approximated formulae.

$\underline{n}$	θ	$\mathrm{Var}\,T$	cost
$(2,1,2,3,1)$	15.55	101.76	21.54
$(2,1,3,3,1)$	14.32	92.84	22.89
$(2,1,3,4,1)$	13.47	84.00	24.28

5. Final Remarks

As one can expect, the method of solving the optimization problem presented in this paper gives better results than the one based on approximated formulae for moments (Krishnan Iyer and Downs, 1978). It is, however, worth stressing that in the numerical example studied we obtained variance which was about 10% less than the one obtained in (Krishnan Iyer and Downs, 1978). The solution with $\underline{n} = (2, 1, 3, 4, 1)$, which is the best in the approximated method (Krishnan Iyer and Downs, 1978), is the 13th in succession when compared with the solutions obtained from the procedure described in Section 3.

Our results contribute also to solving the hypothesis formulated in (Krishnan Iyer and Downs, 1978), i.e. that each time redundancy is added to the system in such a way that the mean lifetime remains unchanged and the total cost does not exceed a fixed value, the standard deviation is reduced. This hypothesis followed from the analysis of numerical results. However, numerical results presented in this paper indicate that the hypothesis is not true. To see this, compare the 2nd and 6th (or 7th) rows in Tab. 1. The system in the 2nd row has a redundant element in unit U_4 (or U_5) and its variance is higher than the variance of the system in the 6th (or 7th) row.

A comparison of exact and approximate approaches to series-parallel systems was given in (Łobos, 2000). The method based on exact formulae for moments gives better results in both cases (elements have constant or random failure rates).

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