

TERNARY WAVELETS AND THEIR APPLICATIONS TO SIGNAL COMPRESSION

GHULAM MUSTAFA*, FALAI CHEN*
ZHANGJIN HUANG*

* Department of Mathematics
University of Science and Technology of China
Hefei, Anhui 230026, P. R. China

e-mail: mustafa_rakib@yahoo.com, {chenfl, zjhuang}@ustc.edu.cn

We introduce ternary wavelets, based on an interpolating 4-point C^2 ternary stationary subdivision scheme, for compressing fractal-like signals. These wavelets are tightly squeezed and therefore they are more suitable for compressing fractal-like signals. The error in compressing fractal-like signals by ternary wavelets is at most half of that given by four-point wavelets (Wei and Chen, 2002). However, for compressing regular signals we further classify ternary wavelets into ‘odd ternary’ and ‘even ternary’ wavelets. Our odd ternary wavelets are better in part for compressing both regular and fractal-like signals than four-point wavelets. These ternary wavelets are locally supported, symmetric and stable. The analysis and synthesis algorithms have linear time complexity.

Keywords: subdivision, wavelets, multiresolution, signal compression

1. Introduction

Multiresolution analysis and wavelets have received considerable attention in recent years. Besides a broad range of applications in approximation theory (Daubechies, 1988), signal processing (Mallat, 1989) and physics, wavelets have also recently been applied to many problems in computer graphics. These graphics applications include image compression (DeVore *et al.*, 1992), fast methods for solving simulation problems in 3D modelling, and animation (Liu *et al.*, 1994), etc. Multiresolution analysis decomposes a complicated function into a low resolution part, together with a collection of perturbations, called wavelets coefficients, necessary to recover the original function. Wavelets provide a powerful and remarkably flexible set of tools for handling fundamental problems in science and engineering. There are many constructions of wavelets for functions parametrized over an interval (Andersson *et al.*, 1993). These have found use in signal processing, signal compression (Wei and Chen, 2002), and many other applications involving functions parametrized in one dimension. Classically, wavelets are functions generated from one basic function by dilations and translations. They admit a hierarchical decomposition. The functions that can be hierarchically decomposed can be generated through a simple process known as recursive subdivision. The strong connection between subdivision and wavelets allows us to create hierarchical bases for multiresolution analysis by subdivision.

In this paper, we develop a new class of wavelets, called ternary wavelets, based on an interpolating 4-point C^2 ternary stationary subdivision scheme (Hassan *et al.*, 2002), and describe how to use them in signal compression. The ternary wavelets are tightly squeezed and therefore are more appropriate for compressing fractal-like signals. Their degree of smoothness depends on various weight parameters. When the values of parameters are large, the corresponding scaling functions and wavelets are fractal-like. The article is organized as follows: First, we present some basic notions required to understand the theoretical frame of subdivision and multiresolution analysis. Secondly, we detail the construction of ternary wavelets based on a modified interpolating 4-point C^2 ternary stationary subdivision scheme. Finally, we give applications of our wavelets in signal compression where we show that our wavelets compare favorably with other similar wavelets, and we summarize the results for future research directions.

2. Subdivision and Multiresolution Analysis

As the present work depends on a previous one, for the clarity of foundations it will be necessary to review the subdivision and multiresolution analysis machinery previously developed (Hassan *et al.*, 2002; Lounsbery *et al.*, 1997).

2.1. An Interpolating 4-Point C^2 Ternary Stationary Subdivision Scheme

Stationary subdivision schemes, which are classified into binary and ternary subdivision schemes, have been studied in the fields of approximation and computer aided geometric design. Their mathematical description over arbitrary control polygons is as follows. Suppose that the initial control points of a control polygon are denoted by $p_i^0, i \in Z$. Then, for binary schemes the refined control points $\{p_i^{j+1}\}$ are obtained recursively from $\{p_i^j\}$ by the following formula:

$$p_i^{j+1} = \sum_{k \in Z} \gamma_{(2k-i)} p_k^j,$$

and similarly for ternary schemes:

$$p_i^{j+1} = \sum_{k \in Z} \gamma_{(3k-i)} p_k^j,$$

where $\gamma = (\gamma_i)$ is the mask of the scheme. There are many binary and ternary univariate subdivision schemes such as a 4-point subdivision scheme (Dyn *et al.*, 1987), a 6-point subdivision scheme (Weissman, 1990), ternary and 3-point univariate subdivision schemes (Hassan and Dodgson, 2001) and an interpolating 4-point C^2 ternary stationary subdivision scheme with a tension parameter (the 4-point ternary scheme, for brevity). We can achieve greater smoothness with the same number of control points by using a ternary rather than a binary subdivision scheme. Also, for the same smoothness, the 4-point ternary scheme has a much smaller support and a slightly lower computational cost than the equivalent binary scheme (Dubuc, 1986). These properties of ternary subdivision schemes motivate us to design wavelets based on a ternary, rather than a binary subdivision scheme. Here we give a brief introduction to the 4-point ternary scheme.

In this scheme, a polygon $P^j = (p_i^j)$ is mapped to a refined polygon $P^{j+1} = (p_i^{j+1})$ by applying the following three-subdivision rules:

$$\begin{cases} p_{3i}^{j+1} = p_i^j, \\ p_{3i+1}^{j+1} = a_0 p_{i-1}^j + a_1 p_i^j + a_2 p_{i+1}^j + a_3 p_{i+2}^j, \\ p_{3i+2}^{j+1} = a_3 p_{i-1}^j + a_2 p_i^j + a_1 p_{i+1}^j + a_0 p_{i+2}^j, \end{cases} \quad (1)$$

where the weights $\{a_j\}$ are given by

$$\begin{cases} a_0 = -\frac{1}{18} - \frac{1}{6}\mu, \\ a_1 = \frac{13}{18} + \frac{1}{2}\mu, \\ a_2 = \frac{7x}{18} - \frac{1}{2}\mu, \\ a_3 = -\frac{1}{18} + \frac{1}{6}\mu, \end{cases}$$

and $a_0 + a_1 + a_2 + a_3 = 1$. However, the above scheme does not interpolate endpoints. We modify it to interpolate the endpoints as follows:

$$\begin{cases} p_1^{j+1} = \frac{3}{4} p_0^j + \frac{1}{4} p_1^j, \\ p_2^{j+1} = \frac{1}{4} p_0^j + \frac{3}{4} p_1^j, \\ p_{3i}^{j+1} = p_i^j, \quad i = 0, 1, \dots, n + 1, \\ p_{3i+1}^{j+1} = a_0 p_{i-1}^j + a_1 p_i^j + a_2 p_{i+1}^j + a_3 p_{i+2}^j, \\ \quad \quad \quad i = 1, 2, \dots, n - 1, \\ p_{3i+2}^{j+1} = a_3 p_{i-1}^j + a_2 p_i^j + a_1 p_{i+1}^j + a_0 p_{i+2}^j, \\ \quad \quad \quad i = 1, 2, \dots, n - 1, \\ p_{3n+1}^{j+1} = \frac{3}{4} p_n^j + \frac{1}{4} p_{n+1}^j, \\ p_{3n+2}^{j+1} = \frac{1}{4} p_n^j + \frac{3}{4} p_{n+1}^j, \end{cases} \quad (2)$$

where the j -th control points are $\{p_i^j\}, i = 0 \rightarrow n + 1$, and the $(j + 1)$ -th control points are $\{p_i^{j+1}\}, i = 0 \rightarrow 3n + 3$.

The smoothness of the limit function f generated by (1) and (2) depends on the tension parameter μ . Generally, f is C^2 for $1/15 < \mu < 1/9$, and it is fractal-like for $\mu > 1/9$. Examples of curves generated by the modified 4-point ternary scheme are shown in Fig. 1.

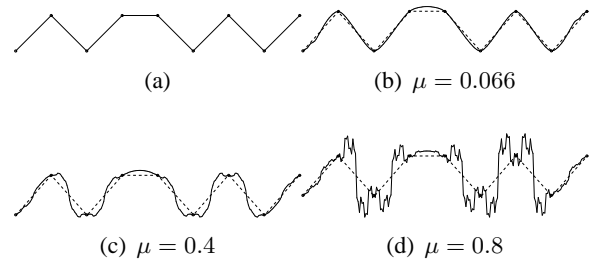


Fig. 1. The modified 4-point ternary scheme for curves: (a) The initial control polygon. (b) The limit curve with $\mu = 0.066$. (c) The limit curve with $\mu = 0.4$. (d) The limit curve with $\mu = 0.8$.

2.2. Multiresolution Analysis

Here we give a brief introduction to the multiresolution analysis construction process. For details, the reader is referred to (Lounsbery *et al.*, 1997; Stollnitz *et al.*, 1996).

The starting point for multiresolution analysis is a nested set of linear function spaces $V^0 \subset V^1 \subset \dots$, with the resolution of functions in V^j increasing with j . These nested spaces can be constructed by considering

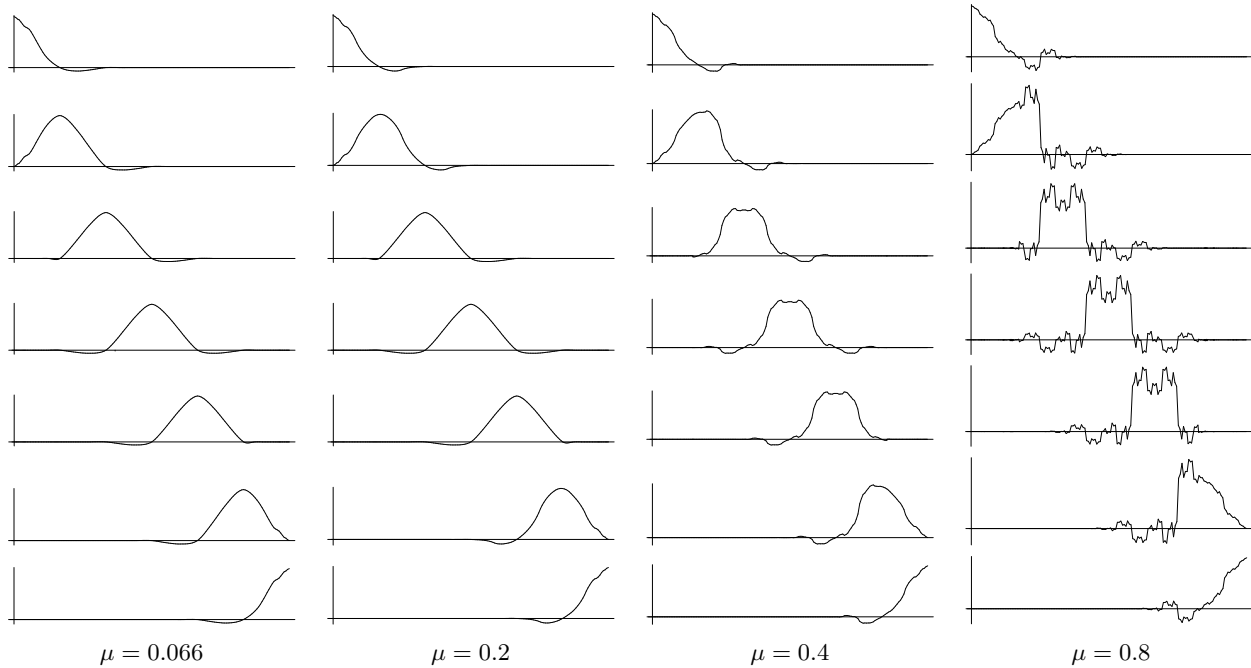


Fig. 2. 4-point ternary scaling functions $\phi_0^j, \phi_1^j, \dots, \phi_n^j$ from top to bottom with different values of the tension parameter.

all linear combinations of translated and scaled functions. Let

$$\Phi^j(x) = \begin{pmatrix} \phi_0^j(x) & \phi_1^j(x) & \dots \end{pmatrix}, \quad j = 0, 1, \dots,$$

be the collection of scaling functions and

$$V^j = \text{span} \{ \phi_0^j(x), \phi_1^j(x), \dots \}.$$

Then nesting these spaces is equivalent to the fact that the scaling functions are refinable, i.e.

$$\Phi^j(x) = \Phi^{j+1}(x)S^j. \tag{3}$$

The next step in multiresolution analysis is to define wavelet spaces also called orthogonal complement spaces, denoted by W^j . The inner product is used to define W^j as

$$W^j = \{ f \in V^{j+1} \mid \langle f, g \rangle = 0, \quad \forall g \in V^j \},$$

where the inner product is

$$\langle f, g \rangle = \int f(x)g(x) dx.$$

The set of functions that span wavelet spaces are called wavelets. The end point for multiresolution analysis is that the analysis and synthesis filters associated with wavelets are constructed and applied in linear time. Mallat (1989) provides a convenient framework to develop the analysis and synthesis filters.

3. Ternary Wavelets

In the previous section, we have introduced the general framework of multiresolution analysis. In the following, our first step is to define the scaling functions for a nested set of function spaces. Then we will construct ternary wavelets.

There is a straightforward recipe for computing scaling functions of ternary wavelets: simply run the modified 4-point ternary scheme starting with a sequence of values $c_k^j = \delta_{i,k}$, $k = 0, 1, \dots, n$ on level j . The limit functions ϕ_i^j are the scaling functions for the ternary wavelets. Figure 2 shows the scaling functions for different tension parameter values. We see that the smoothness of the scaling functions depends on the tension parameter μ when it exceeds 0.11 and then the scaling functions are fractal-like signals. As we know, during the subdivision process by the modified 4-point ternary scheme, at each stage we keep all the old vertices and insert two new vertices “in between” the old ones. The new vertices can be classified into odd and even vertices because these vertices are inserted by using two different subdivision rules. Therefore it is easy to classify the vertices into three categories (old, odd new and even new). If S^j is a subdivision matrix for the modified 4-point ternary scheme and S_O^j , $S_{N_1}^j$ and $S_{N_2}^j$ represent the portions of the subdivision matrix which weight the ‘old’, ‘odd new’ and ‘even new’ vertices, respectively, then S^j can be written down in the

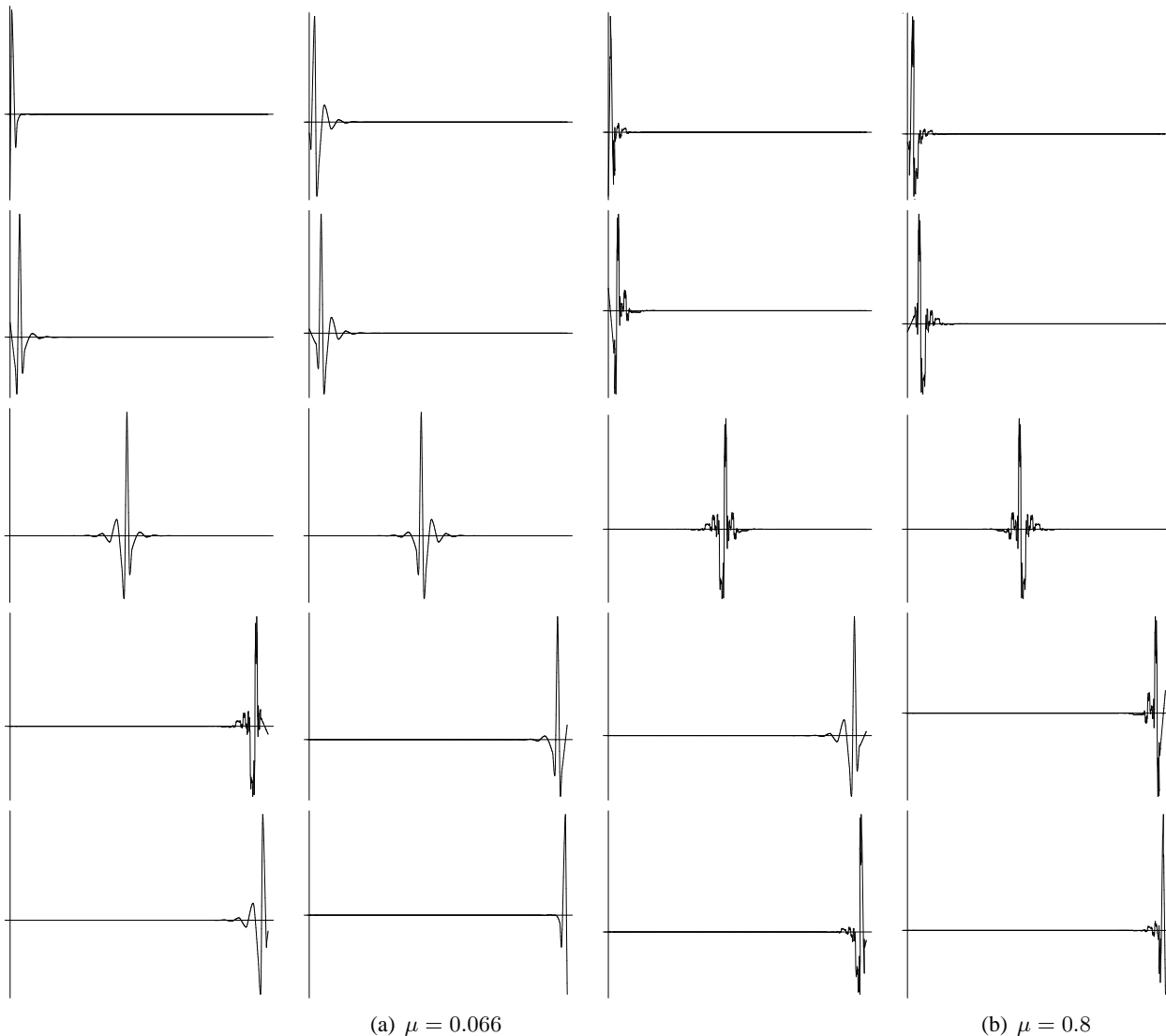


Fig. 3. Ternary wavelets: (a) with $\mu = 0.066$ and (b) $\mu = 0.8$. The first and third columns represent odd ternary wavelets. The second and fourth columns represent even ternary wavelets.

pression is to express an initial set of data using some smaller set of data, either with or without loss of information. Here, we compress some signals and compare the compression results with other existing wavelets.

Signal 1. Given a fractal-like signal generated by the Weierstrass function

$$W(x) = \sum_{j=1}^{\infty} r^{(s-1)j} \sin(r^j x), \quad r > 1,$$

where $s \in (1, 2)$ is the box dimension of $W(x)$, we compress the signal by 4-point wavelets, ternary, even ternary, and odd ternary wavelets. For a fixed compression ratio of 100:18, the L^2 compression errors for 4-point, even ternary, and odd ternary wavelets are 13.30%,

13.60% and 12.79% respectively, but for compression ratio 100:17, the L^2 compression error for ternary wavelets is 6.50%. Figure 4 shows the compression results.

Signal 2. Taking a fractal-like signal, for a fixed compression ratio of 100:3, the compression error for 4-point, even ternary, and odd ternary wavelets are 0.76% , 0.77% and 0.75%, respectively, while the compression error for ternary wavelets is 0.08%. The compressed signals with $w = 0.05$ and $\mu = 0.066$ are shown in Fig. 5.

Signal 3. By taking a regular signal generated by the function

$$F(x) = \sum_{j=1}^{10} r^{(j-r)} \cos(r^{(j-r)} x), \quad r = 1.5,$$

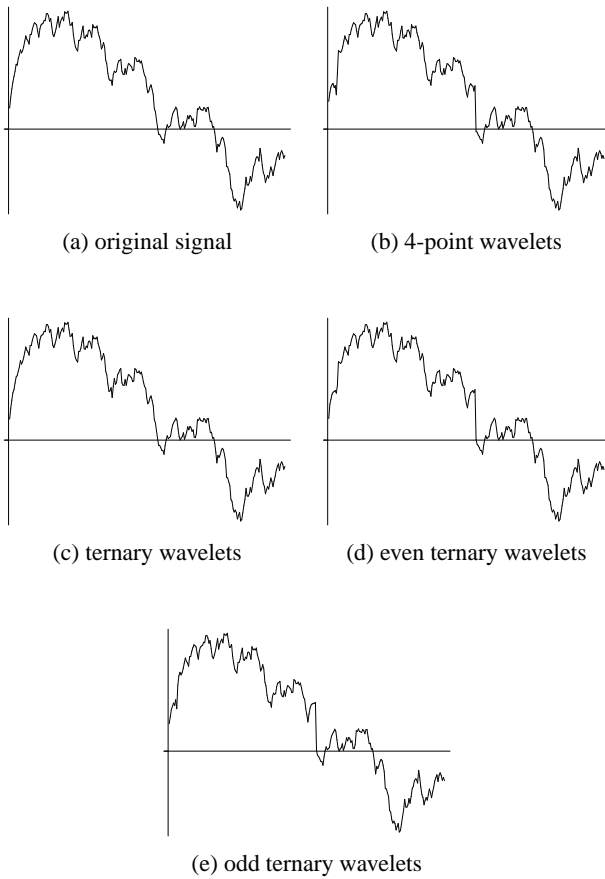


Fig. 4. Compression results: (a) Original signal, $r = 1.5$ and $s = 1.3$ (b) Signal compressed by 4-point wavelets with weight parameter $w = 0.05$, L^2 error being 13.30% (c) Signal compressed by ternary wavelets with tension parameter $\mu = 0.066$, L^2 error being 6.50%. Moreover, (d) and (e) show the compression result by even ternary and odd ternary wavelets with $\mu = 0.066$, L^2 errors being 13.6% and 12.79%, respectively.

we compress it by 4-point, odd ternary, and even ternary wavelets with compression ratio 100:4 and get the penalty of errors 4.35%, 4.25% and 6.21%, respectively. Figure 6 shows the comparison of the compression results.

Signal 4. Here we compress the regular signal 7(a) by 4-point, odd ternary, and even ternary wavelets. The compression errors with compression ratio 100:50 are 0.33%, 0.25% and 0.46%, respectively. The results are shown in Fig. 7.

From the demonstration of Signals 1 and 2, we have shown that the error in compressing fractal-like signals by ternary wavelets is at least half less than that of the error given by 4-point wavelets.

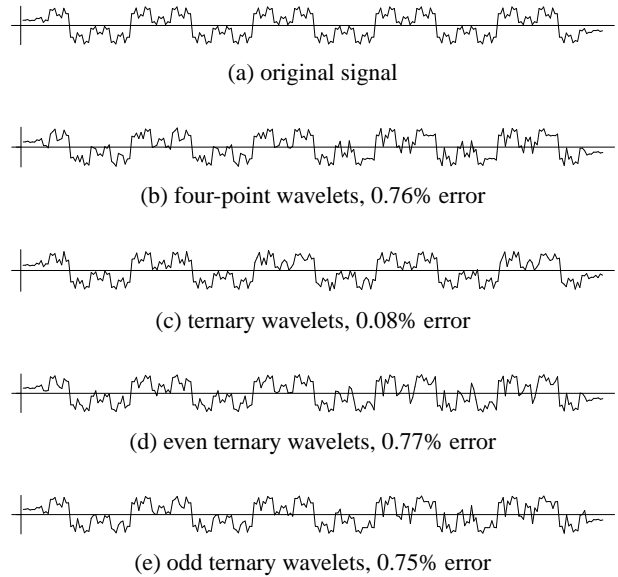


Fig. 5. Signal compression: (a) Original signal (b) Signal compressed by 4-point wavelets, 0.76% level of L^2 error (c) Signal compressed by ternary wavelets, 0.08% level of L^2 error. Moreover, (d) and (e) show the compression results by even ternary and odd ternary wavelets, with errors 0.77% and 0.75%, respectively.

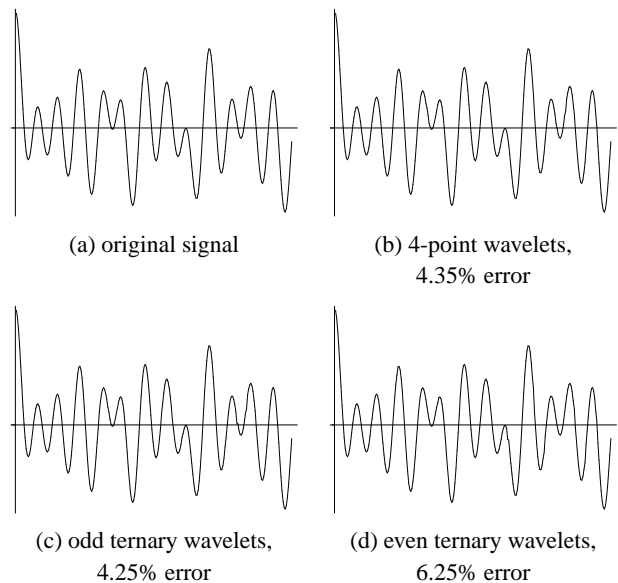


Fig. 6. Compression of a regular signal: (a) Original signal (b) Signal compressed by 4-point wavelets, 4.35% level of error (c) Signal compressed by odd ternary wavelets, 4.25% level of error. Moreover, (d) shows the compression result by even ternary wavelets, level of 6.25%.

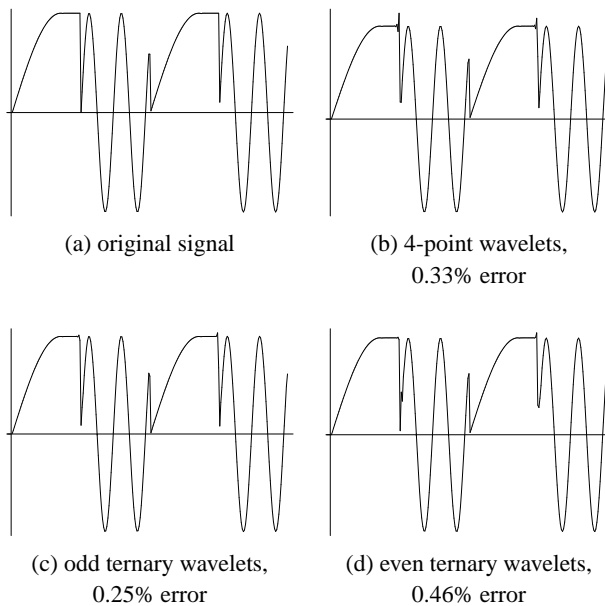


Fig. 7. Compression ratio 100:50: (a) Original signal (b) Compression results by 4-point wavelets, 0.33% level of errors. Moreover, (c) and (d) show compression results by odd ternary and even ternary wavelets with 0.25% and 0.46% errors, respectively.

5. Summary and Future Work

In this paper, based on an interpolating 4-point C^2 ternary stationary subdivision scheme (Hassan *et al.*, 2002), we have introduced ternary wavelets. They are designed for compressing fractal-like signals because of their tightly squeezing property. The error in compressing fractal-like signals by ternary wavelets constitutes at most half of the error produced by 4-point wavelets (Wei and Chen, 2002). However, for compressing regular signals we have further classified these wavelets into ‘odd ternary’ and ‘even ternary’ wavelets. Our odd ternary wavelets are better than 4-point wavelets in the sense of compressing both regular and fractal-like signals. There are still some problems for further investigation such as how to find suitable values of the parameter for compressing particular signals. The generalization of this work to higher dimensions for compressing images, fingerprint compression, denoising images, etc. can be considered as a possible direction of future work. As wavelets are widely used in computer graphics and many other areas, we may investigate some other applications of ternary wavelets in these areas.

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